AN EXTENSION OF THE GENERAL COEFFICIENT THEOREM(1)

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1. In a recent paper [3] we have shown how a sharpening of Teichmüller's coefficient result obtained by a relaxation of the normalization imposed on the mapping function at the single pole of higher order of the quadratic differential involved provides a most effective means for discussing the coefficient problem for univalent functions. In the present paper we will derive the corresponding extension of the General Coefficient Theorem. In addition to those given in [3] this result has a great number of explicit applications to the theory of univalent functions. We will give at the end of this paper one such application to show the additional simplifications which the new result provides.

2. Our present main theorem is enunciated within the same general framework as employed in [2] and standard symbolism and notation will usually be carried over without explicit mention. In particular we refer to [2, Definitions 4.1, 4.2, 4.3] for the concepts *admissible family of domains*, *admissible homotopy into the identity* and *deformation degree*. However it is necessary to extend the definition of an admissible family of functions in the following manner.

DEFINITION 1. *Let* $\{\Delta\}$ *be an admissible family of domains* $\Delta_j, j=1, \cdots, K$, *on the finite oriented Riemann surface* $\mathfrak{R}$ *with respect to the quadratic differential* $Q(z)dz^2$. *Then by an admissible family* $\{f\}$ *of functions* f_j, $j=1, \cdots, K$, *associated with* $\{\Delta\}$ *we mean a family with the following properties*

(i) f_j *maps* Δ_j *conformally into* $\mathfrak{R}, j=1, \cdots, K$,

(ii) *if a pole* A *of* $Q(z)dz^2$ *lies in* $\Delta_j, f_j(A)=A$,

(iii) $f_j(\Delta_j)\cap f_l(\Delta_l)=0, j\neq l, j, l=1, \cdots, K$,

(iv) *if* A *is a pole of order* m *greater than two of* $Q(z)dz^2$ *in* Δ_j, *in terms of a local parameter* z *representing* A *as the point at infinity* $f_j(z)$ *admits locally the representation*

$$f_j(z) = z + \sum_{i=k}^{\infty} \frac{a_i}{z^i} \tag{1}$$

where $m-3\geq k\geq m/2-2$,

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(v) *the family* $\{f\}$ *admits an admissible homotopy* F *into the identity,*

(vi) *the homotopy* F *can be chosen so that if* A *is a pole of order* m *greater than two of* $Q(z)dz^2$ *on the boundary of a strip domain then* $d(F, A)=0$.

In the representation (1) it is understood that the numerical value $f_j(z)$ is that assigned to the function f_j in terms of the local parameter z. If we replace the chosen local parameter z by another admissible parameter $\tilde{z}$ then the expansion of the corresponding function f_j will be given by a series of the same form as (1) although possibly with different numerical values of the coefficients. However we verify that the algebraically largest negative power of z associated with a nonzero coefficient will be the same in each case.

We are now ready to state the extended form of the General Coefficient Theorem.

THEOREM 1. *Let* $\mathcal{R}$ *be a finite oriented Riemann surface,* $Q(z)dz^2$ *a positive quadratic differential on* $\mathcal{R}$, $\{\Delta\}$ *an admissible family of domains* $\Delta_j, j=1, \cdots, K$, *on* $\mathcal{R}$ *relative to* $Q(z)dz^2$ *and* $\{f\}$ *an admissible family of functions* f_j, $j=1, \cdots, K$, *associated with* $\{\Delta\}$. *Let* $Q(z)dz^2$ *have double poles* $P_1, \cdots, P_r$ *and poles* $P_{r+1}, \cdots, P_N$ *of order greater than two. We allow either of these sets to be void but not both. Let* P_j, $j\leqq r$, *lie in the domain* Δ_l *and in terms of a local parameter* z *representing* P_j *as the point at infinity let* f_l *have the expansion*

$$f_l(z) = a^{(j)}z + a_0^{(j)} + \textit{negative powers of } z \tag{2}$$

and Q *the expansion*

$$Q(z) = \alpha^{(j)}z^{-2} + \textit{higher powers of } z^{-1}. \tag{3}$$

Let P_j, $j>r$, *a pole of order* m_j *greater than two, lie in the domain* Δ_l *and in terms of a local parameter* z *representing* P_j *as the point at infinity let* f_l *have the expansion*

$$f_l(z) = z + \sum_{i=k_j}^{\infty} \frac{a_i^{(j)}}{z^i} \tag{4}$$

where k_j *is the smallest integer greater than or equal to* $m_j/2-2$ *and* Q *the expansion*

$$Q(z) = \alpha^{(j)}\left[z^{m_j-4} + \sum_{i=k_j+1}^{\infty} \beta_i^{(j)} z^{m_j-i-4}\right]. \tag{5}$$

Then

$$\mathcal{R}\left\{\sum_{j=1}^{r} \alpha^{(j)} \log a^{(j)} + \sum_{j=r+1}^{N} \alpha^{(j)}\left[a_{m_j-3}^{(j)} + \frac{1}{2}\left(\frac{1}{2}m_j - 2\right)\epsilon_j(a_{k_j}^{(j)})^2 + \epsilon_j\beta_{k_j+1}^{(j)}a_{k_j}^{(j)}\right]\right\} \leqq 0 \tag{6}$$

where $\log a^{(j)} = \log |a^{(j)}| - id(F, P_j)$, $j \leqq r$, *and* $\epsilon_j = 1$ *if* m_j *is even*, $\epsilon_j = 0$ *if* m_j *is odd*, $j > r$.

If equality occurs in (6) *each* f_j, $j = 1, \cdots, K$ *must be an isometry in the Q-metric*

$$|d\zeta| = |Q(z)|^{1/2}|dz|,$$

each trajectory in $\bigcup_{j=1}^{K} \Delta_j$ *must go into another such and the set* $\bigcup_{j=1}^{K} f_j(\Delta_j)$ *must be dense in* $\mathfrak{R}$. *If equality occurs in* (6) f_l *reduces to the identity in a domain* Δ_l *for which any of the following conditions holds.*

(i) *There is in* Δ_l *a pole* P_j, $j > r$, *of order* m_j *such that* $a_i^{(j)} = 0$ *for* $i < m_j - 3$.

(ii) *There is in* Δ_l *a pole* P_j, $j \leqq r$, *with the corresponding coefficient* $a^{(j)}$ *equal to one.*

(iii) *There is in* Δ_l *a simple pole of* $Q(z)dz^2$ *or a point on a trajectory ending in a simple pole.*

Equality can occur in (6) *when there exists a double pole* P_j, $j \leqq r$, *such that for the corresponding coefficient* $|a^{(j)}| \neq 1$ *only when* $\mathfrak{R}$ *is conformally equivalent to the sphere and* $Q(z)dz^2$ *is a quadratic differential whose only critical points are two poles each of order two. If further* $\{\Delta\}$ *consists of a single domain the corresponding function is conformally equivalent to a linear transformation with the points corresponding to these poles as fixed points.*

The weakening of the normalization of the functions f_l at poles of order greater than two allows further possibilities of equality as compared with those in the statement in [2, pp. 51, 52]. Examples can readily be given showing that these can be effectively realized, see [3, Corollaries 2 and 13] and below §10.

REMARK. While the restriction that the expansion (4) should contain no terms in z^{-i} with $0 \leqq i < m_j/2 - 2$ is required by the method employed in the proof of inequality (6) the corresponding restriction on the expansion (5) is imposed only to reduce the complication of the expression forming the second summand in (6). Indeed the latter condition can always be obtained for an arbitrary quadratic differential by a suitable change of the local parameter used for this expansion. Indeed suppose that in terms of the parameter z we have

$$Q(z) = \alpha\left(z^{m-4} + \sum_{i=1}^{\infty} \beta_i z^{m-i-4}\right).$$

Making the change of parameter

$$z = \tilde{z} + c_0 + \sum_{q=1}^{\infty} c_q \tilde{z}^{-q}$$

we obtain the corresponding expression

$$\tilde{Q}(\tilde{z}) = Q(z(\tilde{z}))\left(\frac{dz}{d\tilde{z}}\right)^2$$

$$= \alpha\left[\left(\tilde{z} + c_0 + \sum_{q=1}^{\infty} c_q\tilde{z}^{-q}\right)^{m-4} + \sum_{i=1}^{\infty}\beta_i\left(\tilde{z} + c_0 + \sum_{q=1}^{\infty} c_q\tilde{z}^{-q}\right)^{m-i-4}\right]$$

$$\times\left[1 - \sum_{q=1}^{\infty} qc_q\tilde{z}^{-q-1}\right]^2$$

$$= \alpha\left[\tilde{z}^{m-4} + \sum_{i=1}^{\infty}(\beta_i + (m-4)c_{i-1} + \text{terms in } c_j, j < i-1)\tilde{z}^{m-i-4}\right]$$

$$\times\left[1 - \sum_{q=1}^{\infty}(2qc_q + \text{terms in } c_j, j < q)\tilde{z}^{-q-1}\right]$$

$$= \alpha\left[\tilde{z}^{m-4} + \sum_{i=1}^{\infty}(\beta_i + (m-4-2(i-1))c_{i-1} + \text{terms in } c_j, j < i-1)\tilde{z}^{m-i-4}\right].$$

Thus we can choose the c_j successively to eliminate a finite number of terms in the sum up to the possible point where

$$m - 4 - 2(i-1) = 0$$

that is

$$i = m/2 - 1.$$

Of course this operation entails a corresponding change in the coefficients in the expansion of the function f. Note that if m is odd the above elimination can be continued to any finite number of terms. In any given numerical case it is a simple matter to perform this reduction explicitly. In particular we see that the present enunciation of Theorem 1 immediately implies that previously given for the General Coefficient Theorem.

3. It goes without saying that the proof of the preceding theorem follows the general lines of that used in [2]. However certain arguments must be refined, compare [3], so that we will attempt to give enough detail to make the account readable while referring to [2] for certain steps to avoid repetition.

LEMMA 1. *It is sufficient to prove inequality* (6) *under the further assumption that $Q(z)dz^2$ has no simple poles on* $\mathfrak{R}$.

If $Q(z)dz^2$ has simple poles on $\mathfrak{R}$ since the set H [2, p. 27] is not empty we form a two-sheeted covering surface $\mathfrak{R}^*$ of $\mathfrak{R}$ branched at the simple poles and possibly at one element of H. At points of $\mathfrak{R}^*$ other than these branch points we can use local uniformizing parameters induced by the covering of

$\mathfrak{R}$. At a branch point we can use a local uniformizing parameter Z such that $Z^2=z$ with z a local uniformizing parameter on $\mathfrak{R}$ in terms of which the projection of the branch point is represented by $z=0$. Clearly $\mathfrak{R}^*$ is a finite oriented Riemann surface.

The quadratic differential $Q(z)dz^2$ induces a quadratic differential $Q^*(z)dz^2$ on $\mathfrak{R}^*$ as follows. For a local uniformizing parameter z induced by the covering of $\mathfrak{R}$ we set

$$Q^*(z) = Q(z).$$

For a local uniformizing parameter Z at a branch point we set

$$Q^*(Z) = Q(Z^2)4Z^2.$$

It is then verified [2, p. 52] that at a branch point, corresponding to a simple pole of $Q(z)dz^2$, $Q^*(z)dz^2$ is regular and, corresponding to a double pole of $Q(z)dz^2$, $Q^*(z)dz^2$ has a double pole. The case of a pole P_j with $j>r$ must be checked in more detail owing to the form of the normalization (5). Using now a local parameter Z given by $Z^2=z$ in terms of which the branch point is represented by the point at infinity we have

$$\begin{aligned} Q^*(Z) &= \alpha^{(j)}\left[Z^{2(m_j-4)} + \sum_{i=k_j+1}^{\infty} \beta_i^{(j)} Z^{2(m_j-i-4)}\right]4Z^2 \\ &= 4\alpha^{(j)}\left[Z^{2m_j-6} + \sum_{i=k_j+1}^{\infty} \beta_i^{(j)} Z^{2m_j-2-2i-4}\right] \qquad (7) \\ &= 4\alpha^{(j)}\left[Z^{m_j^*-4} + \sum_{i=k_j^*+1}^{\infty} \beta_i^{*(j)} Z^{m_j^*-i-4}\right] \end{aligned}$$

where $m_j^*=2m_j-2$, $k_j^*=m_j-3$, $\beta_{2i}^{*(j)}=\beta_i^{(j)}$, $\beta_{2i+1}^{*(j)}=0$. We verify directly that the normalization (5) is satisfied.

All remaining conditions and normalizations of the theorem are readily checked. We give explicitly only the verification of the normalization (4) at a pole of order greater than two at a branch point. Using the same local parameter as in the preceding calculation the appropriate function has the expansion

$$Z\left\{1 + \sum_{i=k_j}^{\infty} \frac{a_i^{(j)}}{Z^{2i+2}}\right\}^{1/2} = Z + \sum_{i=k_j^*}^{\infty} \frac{a_i^{*(j)}}{Z^i} \qquad (8)$$

where $a_{2i}^{*(j)}=0$, $i=k_j, \cdots, m_j-3$, $a_{2i+1}^{*(j)}=2^{-1}a_i^{(j)}$, $i=k_j, \cdots, m_j-4$, $a_{2m_j-5}^{*(j)}=2^{-1}a_{m_j-3}^{(j)}-8^{-1}\epsilon_j(a_{k_j}^{(j)})^2$. We see at once that the normalization (4) is satisfied.

It remains to verify that the value of the functional on the left hand side of inequality (6) arising from $\mathfrak{R}^*$ is just twice that for $\mathfrak{R}$. This is immediate for the contributions arising from poles on $\mathfrak{R}$ above which $\mathfrak{R}^*$ is not branched and the corresponding pair of poles on $\mathfrak{R}^*$. In the case of a pole of order two on $\mathfrak{R}$ above which $\mathfrak{R}^*$ is branched the result is verified in [2; p. 54] and for a pole of order greater than two on $\mathfrak{R}$ above which $\mathfrak{R}^*$ is branched it follows from the expansions (7) and (8).

4. The main step in the proof of inequality (6) is now carried out in the usual manner, that is, from the domains in $\{\Delta\}$ we remove suitably fashioned neighborhoods of the points of H. Then in terms of the area of the residual domains in the Q-metric we obtain two evaluations of the area of their images under the functions of $\{f\}$. On the one hand we obtain an estimate from above using the behaviour of the functions of $\{f\}$ on the boundaries of the neighborhoods removed. On the other hand we obtain an estimate from below using the method of the extremal metric.

In the present situation the neighborhoods removed may be chosen in the same manner as in [2, pp. 59, 60]. In each case a simply-connected neighborhood of a pole in H is slit along an open arc on a trajectory or orthogonal trajectory (the latter in the case of a pole of order two in a circle domain) and the residual domain mapped on a portion of Riemann surface by $\zeta=\int(Q(z))^{1/2}dz$. For convenience later on we will assume that for a pole of order greater than two the trajectory used is on the boundary of an end domain at that pole and its image under the chosen branch of $\int(Q(z))^{1/2}dz$ lies above the positive real ζ-axis. For a pole of order greater than two we take the trace on this surface of a square centre the origin of side $2L$ with sides parallel to the coordinate axes. The inverse image of this trace, possibly completed by a trajectory arc forms the boundary of the neighborhood in question. For a pole of order two not in a circle domain we replace the square by the line $\mathfrak{R}\zeta=L$; for a pole of order two in a circle domain by the line $\mathfrak{I}\zeta=L$. In each case we obtain for the pole P_j a closed curve $\gamma(P_j, L)$ which bounds a neighborhood of P_j denoted by $U(P_j, L)$. In general these depend not only on L but on the choice of determination of $\int(Q(z))^{1/2}dz$. However we keep a fixed determination for each P_j. We define

$$\Delta_i(L) = \Delta_i - \bigcup_{j=1}^{N} \overline{U}(P_j, L), \qquad i = 1, \cdots, K$$

and assume L so large that no boundary component of a domain Δ_i meets any $\gamma(P_j, L)$ or penetrates into the interior of any $U(P_j, L)$.

Actually the choice of the figure whose trace we take on the Riemann surface, subject to certain regularity restrictions, is not of too great importance in the present situation. In [3], in order to simplify certain technical considerations, we used a circle rather than a square but the latter would have led to the same results. In general because of their connection with trajectory and orthogonal trajectory structure the figures employed here seem to offer the greatest intuitive naturalness.

5. Let P_j lie in the domain Δ_l. Then for the variable $\zeta=\int(Q(z))^{1/2}dz$ with

the determination chosen above a mapping is induced by the function f_l. In the case of a pole of order greater than two this is done by taking for a point P on the portion of Riemann surface over the ζ-plane the corresponding value z and performing the mapping $f_l(z)$. Providing $|\zeta|$ is large enough for P, $f_l(z)$ will lie in a circular neighborhood of z and we map back by the continuation of $\int(Q(z))^{1/2}dz$ in this neighborhood. We denote by ω the value corresponding to P by this process. It evidently depends in general not only on the value of ζ at P but also on the sheet of the Riemann surface on which P lies. We obtain an expansion for ω in terms of ζ as follows, valid for $|\zeta|$ sufficiently large, where we temporarily omit all subscripts and superscripts j. We have

$$\zeta = \int \alpha^{1/2} z^{m/2-2}\left[1 + \sum_{i=k+1}^{\infty} \beta_i z^{-i}\right]^{1/2} dz$$

$$= \int \alpha^{1/2}\left[z^{m/2-2} + \sum_{i=k+1}^{m-3} \frac{1}{2}\beta_i z^{m/2-i-2} + \left(\frac{1}{2}\beta_{m-2} - \frac{1}{8}\epsilon\beta_{k+1}^2\right) z^{-m/2}\right.$$

$$\left. + \text{ decreasing powers of } z^{1/2}\right] dz$$

$$= \alpha^{1/2}\left(\frac{1}{2}m - 1\right)^{-1} z^{m/2-1} + \frac{1}{2}\epsilon\alpha^{1/2}\beta_{k+1}\log z - (1-\epsilon)\alpha^{1/2}\beta_{k+1}z^{-1/2}$$

$$+ \alpha^{1/2}\sum_{i=k+2}^{m-3}\frac{1}{2}\left(\frac{1}{2}m - i - 1\right)^{-1}\beta_i z^{m/2-i-1}$$

$$+ \alpha^{1/2}\left(-\frac{1}{2}m + 1\right)^{-1}\left(\frac{1}{2}\beta_{m-2} - \frac{1}{8}\epsilon\beta_{k+1}^2\right) z^{-m/2+1}$$

$$+ \text{ decreasing powers of } z^{1/2}$$

where there may be a constant term of integration on the right hand side depending on the choice of determination of $\int(Q(z))^{1/2}dz$. Then for $|\zeta|$ large enough

$$\omega = \alpha^{1/2}\left(\frac{1}{2}m - 1\right)^{-1}\left(z + \sum_{q=k}^{\infty} a_q z^{-q}\right)^{m/2-1} + \frac{1}{2}\epsilon\alpha^{1/2}\beta_{k+1}\log\left(z + \sum_{q=k}^{\infty} a_q z^{-q}\right)$$

$$- (1-\epsilon)\alpha^{1/2}\beta_{k+1}\left(z + \sum_{q=k}^{\infty} a_q z^{-q}\right)^{-1/2}$$

$$+ \alpha^{1/2}\sum_{i=k+2}^{m-3}\frac{1}{2}\left(\frac{1}{2}m - i - 1\right)^{-1}\beta_i\left(z + \sum_{q=k}^{\infty} a_q z^{-q}\right)^{m/2-i-1}$$

$$+ \alpha^{1/2}\left(-\frac{1}{2}m + 1\right)^{-1}\left(\frac{1}{2}\beta_{m-2} - \frac{1}{8}\epsilon\beta_{k+1}^2\right)\left(z + \sum_{q=k}^{\infty} a_q z^{-q}\right)^{-m/2+1} + \cdots.$$

Rearranging we obtain

$$\omega = \zeta + \epsilon\alpha^{1/2}a_k + \sum_{1\leq t<m/2-2} \lambda_t\zeta^{-(m-2t-2)/(m-2)}$$

$$(9)\qquad + \alpha\left(\frac{1}{2}m - 1\right)^{-1}\left[a_{m-3} + \frac{1}{2}\left(\frac{1}{2}m - 2\right)\epsilon a_k^2 + \frac{1}{2}\epsilon\beta_{k+1}a_k\right]\zeta^{-1}$$

$$+ O(|\zeta|^{-1-(m-2)^{-1}})$$

where λ_t are certain constants.

In the case of a pole P_j of order two we take for a point P on the portion of Riemann surface over the ζ-plane the corresponding value z, perform the mapping $f_l(z)$ and map back by the same branch of $\int(Q(z))^{1/2}dz$ following the determination from z to $f_l(z)$ along the reverse of the path curve corresponding to this point under the deformation F. The expansion for ω in terms of ζ is then given by [2, p. 62]

$$(10)\qquad \omega = \zeta + \alpha^{1/2}\log a + O(e^{-|\zeta|})$$

where we have omitted superscripts j appertaining to the particular pole and $\log a$ is the same determination of the logarithm indicated in the statement of Theorem 1.

6. Let us now denote $f_i(\Delta_i(L))$ by $\Delta_i'(L)$. We will estimate the area of $\bigcup_{i=1}^K \Delta_i'(L)$ in the Q-metric from above in terms of the area of $\bigcup_{i=1}^K \Delta_i(L)$ also in the Q-metric. It is seen at once that each of these areas is finite. We observe that the area of $\bigcup_{i=1}^K \Delta_i'(L)$ is bounded above by the area of the domain on $\mathfrak{R}$ having as boundaries the image curves of the curves $\gamma(P_j, L), j=1, \cdots, N$. Since the area of $\bigcup_{i=1}^K \Delta_i(L)$ is equal to the area of the domain bounded by the curves $\gamma(P_j, L)$ it is enough to determine the change in area arising from the displacement of each such boundary curve under its mapping by the appropriate function in $\{f\}$.

Let first $P_j \in H$, $j>r$, be a pole of $Q(z)dz^2$ of order m_j greater than two lying in Δ_l. Under the mapping $\zeta = \int(Q(z))^{1/2}dz$ there corresponds to $\gamma(P_j, L)$ a finite sequence of segments differing from m_j-2 half-boundaries of squares (centre the origin and of side $2L$) by at most one vertical and one horizontal segment, each of length independent of L. The change in area may be regarded as arising from the effect of the mapping (9) on m_j-2 consecutive half-boundaries of squares adjusted by the effect on possible horizontal and vertical segments where the vertical segment may have to be added or subtracted.

The former contribution is seen at once to be

$$\mathfrak{R}\left\{\frac{1}{2i}\int\left(\bar\zeta + \bar\kappa + \sum_{1\leq t<m/2-2}\bar\lambda_t\bar\zeta^{-(m-2t-2)/(m-2)} + \bar\mu\bar\zeta^{-1}\right)\right.$$

$$\left.\times\left(1 + \sum_{1\leq t<m/2-2}\left(-\frac{m-2t-2}{m-2}\right)\lambda_t\zeta^{-1-(m-2t-2)/(m-2)} - \mu\zeta^{-2}\right)d\zeta - \frac{1}{2i}\int\bar\zeta d\zeta\right\}$$

$$+ O(L^{-(m-2)^{-1}})$$

where, in terms of the simplified notation of formula (9), we have written κ for $\epsilon\alpha^{1/2}a_k$, μ for $\alpha(m/2-1)^{-1}[a_{m-3}+2^{-1}(m/2-2)\epsilon a_k^2+2^{-1}\epsilon\beta_{k+1}a_k]$ and the integral is taken over $m-2$ consecutive half-boundaries of squares. A direct calculation shows that this expression is $O(L^{-(m-2)^{-1}})$.

Unless $\epsilon=1$ in the expansion (9) the change in area arising from possible horizontal and vertical segments is also $O(L^{-(m-2)^{-1}})$. However if $\epsilon=1$ we must have m even, thus a circuit about the point at infinity in the z-sphere returns us in the Riemann surface over the ζ-plane to a point whose affix has been translated by $\pi i\epsilon\alpha^{1/2}\beta_{k+1}$. Since up to terms which are $O(L^{-(m-2)^{-1}})$ the transformation from ζ to ω is a translation through $\epsilon\alpha^{1/2}a_k$ the contribution from the segments is seen to be

$$\mathcal{I}\{\pi i\epsilon\alpha^{1/2}\beta_{k+1}\epsilon\bar{\alpha}^{1/2}\bar{a}_k\} + O(L^{-(m-2)^{-}}) = \Re\{\pi\epsilon\,|\,\alpha\,|\,\beta_{k+1}\bar{a}_k\} + O(L^{-(m-2)^{-1}}).$$

Thus the total change in area arising from the effect of f_l on $\gamma(P_j, L)$, $j>r$, is

$$\Re\{\pi\epsilon_j\,|\,\alpha^{(j)}\,|\,\beta^{(j)}_{k_j+1}\bar{a}^{(j)}_{k_j}\} + o(1)$$

where we have resumed our complete notation.

Next let $P_j\in H$, $j\leqq r$, be a pole of order two lying in Δ_l. Then it is verified as in [2, pp. 61, 62] that the change in area arising from the effect of f_l on $\gamma(P_j, L)$ is

$$2\pi\Re\{\,|\,\alpha^{(j)}\,|\,\log a^{(j)}\} + o(1).$$

Combining these results we have our desired estimate from above

$$\begin{aligned}\sum_{i=1}^{K}\iint_{\Delta_i'(L)} dA \leqq \sum_{i=1}^{K}\iint_{\Delta_i(L)} dA &+ \sum_{j=1}^{r} 2\pi\Re\{\,|\,\alpha^{(j)}\,|\,\log a^{(j)}\} \\ &+ \sum_{j=r+1}^{N}\pi\Re\{\epsilon_j\,|\,\alpha^{(j)}\,|\,\beta^{(j)}_{k_j+1}\bar{a}^{(j)}_{k_j}\} + o(1).\end{aligned} \tag{11}$$

In this expression dA denotes the element of area in the Q-metric.

7. Next we estimate the area of $\bigcup_{i=1}^{K}\Delta_i'(L)$ (in the Q-metric) from below in terms of the area of $\bigcup_{i=1}^{K}\Delta_i(L)$ by the use of the method of the extremal metric. This time we interpret the former quantity as the area of $\bigcup_{i=1}^{K}\Delta_i(L)$ in a new metric. For $P\in\Delta_i$ we denote by $|f_i'(P)|$ the distortion produced at the point P by the mapping f_i relative to the Q-metric. We now define the metric $\rho|d\zeta|$ on $\bigcup_{i=1}^{K}\Delta_i$ by

$$\rho(P)\,|\,d\zeta(P)\,| = |\,f_i'(P)\,|\,|\,d\zeta(P)\,|, \qquad P\in\Delta_i, \qquad i=1,\cdots,K$$

where $|d\zeta| = |Q(z)|^{1/2}|dz|$. Then it is clear that

$$\iint_{\bigcup_{i=1}^{K}\Delta_i(L)}\rho^2 dA = \sum_{i=1}^{K}\iint_{\Delta_i'(L)} dA.$$

For each type of basic domain associated with the trajectory structure of

$Q(z)dz^2$ on $\mathfrak{R}$ as in [2, Theorem 3.5] we now take the intersection with $\bigcup_{i=1}^{K} \Delta_i(L)$ and estimate the area of this intersection in the metric $\rho|d\zeta|$.

If we form the closure of all trajectories of $Q(z)dz^2$ which have a limiting end point at a zero of $Q(z)dz^2$ (normally we would include simple poles but at the moment we are assuming they are not present) the interior of this set consists of a finite number of domains on $\mathfrak{R}$ whose union we denote by $\hat{\Phi}$. Denoting $\hat{\Phi} \cap \bigcup_{i=1}^{K} \Delta_i$ by Φ^* we know that

$$\Phi^* \subset \bigcup_{i=1}^{K} \Delta_i(L)$$

for all admissible L. Then we verify as in [2, pp. 63, 64] that

$$\iint_{\Phi^*} \rho^2 dA \geqq \iint_{\Phi^*} dA. \tag{12}$$

Next let $\mathfrak{D}$ be a ring domain in the trajectory structure of $Q(z)dz^2$. Then we denote $\mathfrak{D} \cap \bigcup_{i=1}^{K} \Delta_i$ by $\mathfrak{D}^*$ and observe that

$$\mathfrak{D}^* \subset \bigcup_{i=1}^{K} \Delta_i(L)$$

for all admissible L. As in [2, p. 64] we see that

$$\iint_{\mathfrak{D}^*} \rho^2 dA \geqq \iint_{\mathfrak{D}^*} dA. \tag{13}$$

Let $\mathfrak{C}$ be a circle domain in the trajectory structure of $Q(z)dz^2$. We denote by $\mathfrak{C}(L)$ the intersection of $\mathfrak{C}$ with $\bigcup_{i=1}^{K} \Delta_i(L)$. Then in the same manner we have

$$\iint_{\mathfrak{C}(L)} \rho^2 dA \geqq \iint_{\mathfrak{C}(L)} dA. \tag{14}$$

Consider now a pole P_j, $j > r$, of order m_j greater than two lying say in the domain Δ_i of the family $\{\Delta\}$. With it are associated $m_j - 2$ end domains which we denote by $\mathcal{E}_1, \cdots, \mathcal{E}_{m_j-2}$ taken in cyclic order starting with the trajectory used to slit the simply-connected neighborhood of P_j in §4 and possible strip domains. Each $\mathcal{E}_q$, $q = 1, \cdots, m_j - 2$, meets no domain in $\{\Delta\}$ other than Δ_i. We denote $\mathcal{E}_q \cap \Delta_i(L)$ by $\mathcal{E}_q(L)$. Then by the assigned branch of $\int (Q(z))^{1/2} dz$, $\mathcal{E}_1$ is mapped onto an upper half-plane and the remaining end domains alternately on lower and upper half-planes. Further $\mathcal{E}_1(L)$ is mapped onto a half-square $E_1(L)$

$$-L < \xi < L, \qquad 0 < \eta < L \qquad\qquad (\zeta = \xi + i\eta)$$

provided perhaps with a finite number of horizontal slits. As in [2, p. 64] we take the union $\mathcal{E}^*$ of $\mathcal{E}_1$ with two consecutive end domains in the trajectory structure of $-Q(z)dz^2$ (on a covering surface of $\mathfrak{R}$ if necessary) so that $\mathcal{E}^*$ is mapped by the chosen branch of $\int(Q(z))^{1/2}dz$ onto the union E^* of an upper half-plane, a right hand half-plane and a left hand half-plane. We transfer the metric $\rho(P)|d\zeta(P)|$ to the ζ-plane by setting

$$\rho(\zeta)\,|\,d\zeta\,| = \rho(P)\,|\,d\zeta(P)\,|.$$

No confusion arises by using the notation ρ in each case. Clearly

$$\iint_{\mathcal{E}_1(L)} \rho dA = \iint_{E_1(L)} \rho d\xi d\eta.$$

Let now $\sigma(\eta)$ denote the horizontal segment of length $2L$ lying in $E_1(L)$ on the line $\mathfrak{I}\zeta=\eta$. This will exist for all but a finite number of η in $(0, L)$. Let $A(\eta)$, $B(\eta)$ be the end points of $\sigma(\eta)$ on the lines $\xi=L$, $\xi=-L$ respectively. Let ϕ_i be the mapping induced on E^* by f_i as in §5. Then by the exact argument used in [2, p. 65] we find that

$$\int_{\sigma(\eta)} \rho d\xi \geqq \mathfrak{R}[\phi_i(A(\eta)) - \phi_i(B(\eta))].$$

Using the expansion (9) and writing μ for

$$\alpha(m/2-1)^{-1}[a_{m-3}+2^{-1}(m/2-2)\epsilon a_k^2+2^{-1}\epsilon\beta_{k+1}a_k]$$

while setting $\eta=L\tan\theta$ we have

$$\int_{\sigma(\eta)} \rho d\xi \geqq 2L + \sum_{1\leq t<m/2-2} \mathfrak{R}\left\{\lambda_t\,|\,\zeta\,|^{-(m-2t-2)/(m-2)} \times \left(\exp\left(-\frac{m-2t-2}{m-2}\,i\theta\right) - \exp\left(-\frac{m-2t-2}{m-2}\,i(\pi-\theta)\right)\right)\right\} + 2(\mathfrak{R}\mu)L^{-1}\cos^2\theta + O(L^{-1-(m-2)^{-1}}).$$

Integrating this with respect to η over the range $0<\eta<L$ we have (neglecting as we may the finite number of values of η for which $\sigma(\eta)$ may not be defined)

$$\iint_{E_1(L)} \rho d\xi d\eta \geqq \iint_{E_1(L)} d\xi d\eta + L \sum_{1\leq t<m/2-2} \mathfrak{R}\left\{\lambda_t \int_0^{\pi/4} |\,\zeta\,|^{-(m-2t-2)/(m-2)} \times \left(\exp\left(-\frac{m-2t-2}{m-2}\,i\theta\right) - \exp\left(-\frac{m-2t-2}{m-2}\,i(\pi-\theta)\right)\right)\sec^2\theta d\theta\right\} + \frac{\pi}{2}\,\mathfrak{R}\mu + O(L^{-(m-2)^{-1}}).$$

We may write instead

$$\iint_{\mathcal{E}_1(L)} \rho dA \geqq \iint_{\mathcal{E}_1(L)} dA + L \sum_{1\leq t<m/2-2} \Re \left\{ \lambda_t \int_0^{\pi/4} |\zeta|^{-(m-2t-2)/(m-2)} \right.$$
$$\left. \times \left(\exp\left(-\frac{m-2t-2}{m-2} i\theta \right) - \exp\left(-\frac{m-2t-2}{m-2} i(\pi-\theta) \right) \right) \sec^2 \theta d\theta \right\}$$
$$+ \frac{\pi}{2} \Re\mu + O(L^{-(m-2)^{-1}}).$$

Similarly we have for $q=2, \cdots, m-2$

$$\iint_{\mathcal{E}_q(L)} \rho dA \geqq \iint_{\mathcal{E}_q(L)} dA + (-1)^{q-1} L \sum_{1\leq t<m/2-2} \Re \left\{ \lambda_t \int_0^{\pi/4} |\zeta|^{-(m-2t-2)/(m-2)} \right.$$
$$\times \left(\exp\left(-\frac{m-2t-2}{m-2} i((q-1)\pi+\theta) \right) \right.$$
$$\left. \left. - \exp\left(-\frac{m-2t-2}{m-2} i(q\pi-\theta) \right) \right) \sec^2 \theta d\theta \right\}$$
$$+ \frac{\pi}{2} \Re\mu + O(L^{-(m-2)^{-1}}).$$

Adding up all these terms we have

$$\sum_{q=1}^{m-2} \iint_{\mathcal{E}_q(L)} \rho dA \geqq \sum_{q=1}^{m-2} \iint_{\mathcal{E}_q(L)} dA + (m-2) \frac{\pi}{2} \Re\mu$$
$$+ L \sum_{q=1}^{m-2} (-1)^{q-1} \sum_{1\leq t<m/2-2} \Re \left\{ \lambda_t \int_0^{\pi/4} |\zeta|^{-(m-2t-2)/(m-2)} \right.$$
$$\times \left(\exp\left(-\frac{m-2t-2}{m-2} i((q-1)\pi+\theta) \right) \right.$$
$$\left. \left. - \exp\left(-\frac{m-2t-2}{m-2} i(q\pi-\theta) \right) \right) \sec^2 \theta d\theta \right\}$$
$$+ O(L^{-(m-2)^{-1}}).$$

The third term on the right hand side can be written as

$$L \sum_{1\leq t<m/2-2} \Re \left\{ \lambda_t \int_0^{\pi/4} |\zeta|^{-(m-2t-2)/(m-2)} \left(\exp\left(-\frac{m-2t-2}{m-2} i\theta \right) \right. \right.$$
$$\left. + \exp\left(\frac{m-2t-2}{m-2} i\theta \right) \right)$$
$$\times \left(\sum_{q=1}^{m-2} (-1)^{q-1} \exp\left(-\frac{m-2t-2}{m-2} (q-1)\pi i \right) \right) \sec^2 \theta d\theta$$

but

$$\sum_{q=1}^{m-2} (-1)^{q-1} \exp\left(-\frac{m-2t-2}{m-2}(q-1)\pi i\right) = 0$$

so that the whole expression is zero.

The use of a familiar inequality now gives

$$\sum_{q=1}^{m-2} \iint_{\mathcal{E}_q(L)} \rho^2 dA \geqq \sum_{q=1}^{m-2} \iint_{\mathcal{E}_q(L)} dA + \pi(m-2)\Re\mu + O(L^{-(m-2)^{-1}}).$$

Summing these results over all poles of order greater than two we have ultimately

$$\sum \iint_{\mathcal{E}(L)} \rho^2 dA \geqq \sum \iint_{\mathcal{E}(L)} dA + 2\pi\Re\left\{\sum_{j=r+1}^{N} \alpha^{(j)} \times \left[a_{m_j-3}^{(j)} + \frac{1}{2}\left(\frac{1}{2}m_j - 2\right)\epsilon_j (a_{k_j}^{(j)})^2 + \frac{1}{2}\epsilon_j \beta_{k_j+1}^{(j)} a_{k_j}^{(j)}\right]\right\} + o(1) \tag{15}$$

where the first two summations are each taken over the totality of end domains.

Finally let $\mathcal{S}$ be a strip domain which has boundary elements arising from poles P_q and P_l which may be distinct or coincident and of any order greater than or equal to two. These poles must lie in the same domain, say Δ_i, of the family $\{\Delta\}$ and $\mathcal{S}$ can meet no domain in $\{\Delta\}$ other than Δ_i. We denote $\mathcal{S}\cap\Delta_i(L)$ by $\mathcal{S}(L)$. A suitable determination of $\zeta = \int (Q(z))^{1/2} dz$ maps $\mathcal{S}$ onto a strip S given by

$$0 < \Im\zeta < \lambda$$

(λ positive) where the boundary element of $\mathcal{S}$ arising from P_q corresponds to the boundary point of S at infinity in whose neighborhood $\Re\zeta$ becomes positively infinite and the boundary element of $\mathcal{S}$ arising from P_l corresponds to the boundary point of S at infinity in whose neighborhood $\Re\zeta$ becomes negatively infinite. As in [2, pp. 66, 67] we form the union S^* of S and two half-planes containing respective right and left half-strips of S (lying on a Riemann covering surface if necessary). Under our chosen determination of $\int (Q(z))^{1/2} dz$ to $\mathcal{S}(L)$ corresponds a rectangle $S(L)$

$$-L + b < \xi < L + a, \qquad 0 < \eta < \lambda \qquad (\zeta = \xi + i\eta)$$

provided perhaps with a finite number of horizontal slits where a, b are real numbers possibly positive or negative (since the present determination may not be the one used to define the neighborhoods $U(P_j, L)$). As before we transfer the metric $\rho(P)|d\zeta(P)|$ to the ζ-plane by setting

$$\rho(\zeta)\,|\,d\zeta\,| = \rho(P)\,|\,d\zeta(P)\,|$$

and we see at once

$$\iint_{\mathcal{S}(L)} \rho dA = \iint_{S(L)} \rho d\xi d\eta.$$

Let $\sigma(\eta)$ denote the horizontal segment of length $2L+a-b$ lying in $S(L)$ on the line $\mathfrak{s}(\zeta)=\eta$ which exists for all but at most a finite number of η in $0<\eta<\lambda$. Let $A(\eta)$, $B(\eta)$ be the end points of $\sigma(\eta)$ on the lines $\xi=L+a$, $\xi=-L+b$ respectively. Let us denote by ϕ_{ij}, $j=q, l$, the mapping induced by f_i on S^* according to the prescription given in §5 pertaining to the pole P_j. The current determination ζ_2 of $\int(Q(z))^{1/2}dz$ is related to the determination ζ_1 used to define the neighborhood $U(P_j, L)$, $j=q, l$, by a relation $\zeta_2=c\pm\zeta_1$ where c is a constant. Let $\delta_j=\pm 1$ according as we have $\pm\zeta_1$. Then if P_j, $j=q, l$, is a pole of order greater than two the mapping $\omega=\phi_{ij}(\zeta)$ is represented at the respective points $A(\eta)$ and $B(\eta)$ by the expansion

$$\omega = \zeta + \delta_j\epsilon_j(\alpha^{(j)})^{1/2}a_{k_j}^{(j)} + O(L^{-(m-2)^{-1}}).$$

If P_j is a pole of order two the mapping is represented by the expansion

$$\omega = \zeta + \delta_j(\alpha^{(j)})^{1/2}\log a^{(j)} + o(1)$$

(note that in this case $\delta_q=1$, $\delta_l=-1$) where $(\alpha^{(j)})^{1/2}=\left|\alpha^{(j)}\right|^{1/2}e^{i\psi_j}$ is the root with positive real part and $\log a^{(j)}$ has the determination given in the statement of Theorem 1. We denote this expansion generically by

$$\omega = \zeta + \tau_j + o(1).$$

As in [2, pp. 67, 68] we verify that $\int_{\sigma(\eta)}\rho d\xi \geqq \Re[\phi_{iq}(A(\eta))-\phi_{il}(B(\eta))]$. Thus

$$\int_{\sigma(\eta)} \rho d\xi \geqq 2L + a - b + \Re\tau_q - \Re\tau_l + o(1).$$

Integrating this with respect to η over the range $0<\eta<\lambda$ (since we may neglect the finite number of values of η for which $\sigma(\eta)$ is not defined) we get

$$\iint_{S(L)} \rho d\xi d\eta \geqq \lambda[2L + a - b] + \lambda\Re\tau_q - \lambda\Re\tau_l + o(1)$$

which may be written as

$$\iint_{\mathcal{S}(L)} \rho dA \geqq \iint_{\mathcal{S}(L)} dA + \lambda\Re\tau_q - \lambda\Re\tau_l + o(1).$$

Applying the familiar inequality gives

$$\iint_{\mathcal{S}(L)} \rho^2 dA \geqq \iint_{\mathcal{S}(L)} dA + 2\lambda\Re\tau_q - 2\lambda\Re\tau_l + o(1).$$

Summing over all strip domains we finally have

$$\sum \iint_{\mathcal{S}(L)} \rho^2 dA \geqq \sum \iint_{\mathcal{S}(L)} dA + 4\pi \Re \left\{ \sum_{j=1}^{r} \left| \alpha^{(j)} \right| e^{i\psi_j} \cos \psi_j \log a^{(j)} \right\} \tag{16}$$
$$+ 2 \sum_{j=r+1}^{N} \Re\left\{ \pi \epsilon_j (\alpha^{(j)})^{1/2} \beta_{k_j+1}^{(j)} \right\} \Re\left\{ \epsilon_j (\alpha^{(j)})^{1/2} a_{k_j}^{(j)} \right\} + o(1)$$

where for a pole of order two in a circle domain we understand that $\psi_j = \pi/2$ and the first two summations are taken over the totality of strip domains.

Adding inequalities (12), (13), (14), (15), (16) for all domains into which $\bigcup_{i=1}^{K} \Delta_i(L)$ is decomposed we have the desired evaluation from below

$$\sum_{i=1}^{K} \iint_{\Delta_i(L)} \rho^2 dA \geqq \sum_{i=1}^{K} \iint_{\Delta_i(L)} dA + 4\pi \Re \left\{ \sum_{j=1}^{r} \left| \alpha^{(j)} \right| e^{i\psi_j} \cos \psi_j \log a^{(j)} \right\}$$
$$+ 2\pi \Re \left\{ \sum_{j=r+1}^{N} \alpha^{(j)} \left[a_{m_j-3}^{(j)} + \frac{1}{2}\left(\frac{1}{2} m_j - 2 \right) \epsilon_j (a_{k_j}^{(j)})^2 + \frac{1}{2} \epsilon_j \beta_{k_j+1}^{(j)} a_{k_j}^{(j)} \right] \right\} \tag{17}$$
$$+ 2\pi \sum_{j=r+1}^{N} \Re\left\{ \epsilon_j (\alpha^{(j)})^{1/2} \beta_{k_j+1}^{(j)} \right\} \Re\left\{ \epsilon_j (\alpha^{(j)})^{1/2} a_{k_j}^{(j)} \right\} + o(1).$$

8. Recalling that

$$\sum_{i=1}^{K} \iint_{\Delta_i'(L)} dA = \sum_{i=1}^{K} \iint_{\Delta_i(L)} \rho^2 dA$$

and combining inequalities (11) and (17) we obtain

$$2\pi \Re \left\{ \sum_{j=1}^{r} \left| \alpha^{(j)} \right| \log a^{(j)} \right\} + \pi \sum_{j=r+1}^{N} \Re\left\{ \epsilon_j \left| \alpha^{(j)} \right| \beta_{k_j+1}^{(j)} \bar{a}_{k_j}^{(j)} \right\} + o(1)$$
$$\geqq 4\pi \Re \left\{ \sum_{j=1}^{r} \left| \alpha^{(j)} \right| e^{i\psi_j} \cos \psi_j \log a^{(j)} \right\}$$
$$+ 2\pi \Re \left\{ \sum_{j=r+1}^{N} \alpha^{(j)} \left[a_{m_j-3}^{(j)} + \frac{1}{2}\left(\frac{1}{2} m_j - 2 \right) \epsilon_j (a_{k_j}^{(j)})^2 + \frac{1}{2} \epsilon_j \beta_{k_j+1}^{(j)} a_{k_j}^{(j)} \right] \right.$$
$$+ 2\pi \sum_{j=r+1}^{N} \Re\left\{ \epsilon_j (\alpha^{(j)})^{1/2} \beta_{k_j+1}^{(j)} \right\} \Re\left\{ \epsilon_j (\alpha^{(j)})^{1/2} a_{k_j}^{(j)} \right\} + o(1).$$

The explicit terms are independent of L and an elementary calculation gives

$$\Re \left\{ \sum_{j=1}^{r} \alpha^{(j)} \log a^{(j)} + \sum_{j=r+1}^{N} \alpha^{(j)} \left[a_{m_j-3}^{(j)} + \frac{1}{2}\left(\frac{1}{2} m_j - 2 \right) \epsilon_j (a_{k_j}^{(j)})^2 \right. \right.$$
$$\left. \left. + \epsilon_j \beta_{k_j+1}^{(j)} a_{k_j}^{(j)} \right] \right\} \leqq 0.$$

This completes the proof of inequality (6).

9. As regards the question of equality in (6) we see that this can occur only if it does in each of the inequalities (12), (13), (14), (15), (16). It should be pointed out that so far this applies to the problem as simplified in Lemma 1. Equality in these is possible only if $\rho(P)\equiv 1$ in $\bigcup_{i=1}^{K}\Delta_i - H$, that is, the mappings by the functions in $\{f\}$ are isometric in the Q-metric. Further each trajectory in an end, strip or circle domain must be mapped again into a trajectory by the appropriate function in $\{f\}$ otherwise an additional positive additive term would appear on the right hand side of (14), (15) or (16). There is at least one such domain in the trajectory structure of $Q(z)dz^2$ under the assumptions of Theorem 1. Further there can be no open set in the complement of $\bigcup_{i=1}^{K} f_i(\Delta_i)$ in order that equality obtain in (11). Thus every trajectory of $Q(z)dz^2$ in $\bigcup_{i=1}^{K}\Delta_i$ must be carried into another such. It is clear that these conclusions remain valid even when the simplifying assumption of Lemma 1 is dropped.

The remaining equality statements of Theorem 1 come under the earlier more special case of the General Coefficient Theorem and are proved as in [2, pp. 69, 70].

Finally it should be remarked that there exists an extension of [2, Theorem 4.2] to the present more general situation. The reader will easily supply the formal statement.

10. The present extended form of the General Coefficient Theorem has many interesting explicit applications. All the results of [3] come under this heading. We want to give here one other application to show the operation of the present theorem.

We begin by giving the mappings which play the role of extremal mappings in our problem. They are closely related to certain functions given elsewhere [2, pp. 127–129] but it is desirable to discuss them independently here in order to be able to treat the case of equality in the succeeding theorem in complete detail.

LEMMA 2. *Let $Q_\nu(w)dw^2$ denote the quadratic differential*

$$(w^2 - \nu^2)dw^2/w^2$$

where $\nu>0$. Then $Q_\nu(w)dw^2$ has two trajectories T_ν, T_ν' each of which joins the points $\pm\nu$. For $\nu\leqq 4/\pi$ there exists a function $f_\nu\in\Sigma$ such that the mapping $w=f_\nu(z)$ carries E^*: $|z|>1$ onto a domain bounded by T_ν, T_ν' and possible slits of equal length on the real axis to the right of ν and the left of $-\nu$. The expansion of $f_\nu(z)$ about the point at infinity begins

$$z+\frac{1}{2}(2-\nu^2)/z+\cdots. \tag{18}$$

The interior of the complement of $f_\nu(E^*)$ has inner conform radius with respect to the origin equal to

$$2\nu e^{-1}. \tag{19}$$

For $\nu<4/\pi$ the functions in Σ which can be obtained from $f_\nu(z)$ by translation along the trajectories of $Q_\nu(w)dw^2$ make up a 1-parameter family $F_\nu(z, s)$, $-(2-\pi\nu/2)\leqq s\leqq 2-\pi\nu/2$, where $F_\nu(z, s)$ is obtained from $f_\nu(z)$ by a translation of amount s along the trajectories of $Q_\nu(w)dw^2$. The expansion of $F_\nu(z, s)$ about the point at infinity begins

$$z + s + \frac{1}{2}(2 - \nu^2)/z + \cdots. \tag{20}$$

For $\nu>4/\pi$ there exists a function $f_\nu\in\Sigma$ such that the mapping $w=f_\nu(z)$ carries E^* onto a domain bounded by a closed trajectory of $Q_\nu(w)dw^2$ which separates T_ν, T_ν' from the origin. The expansion of $f_\nu(z)$ about the point at infinity begins

$$z + \left(2k^{-2} - 1 - \frac{1}{2}\nu^2\right)\Big/ z + \cdots \tag{21}$$

where

$$\frac{2}{k}E(k) = \frac{1}{2}\pi\nu.$$

The interior of the complement of $f_\nu(E^*)$ has inner conform radius with respect to the origin equal to

$$2\nu e^{-1}\exp\left[-2k^{-1}\nu^{-1}[K'(k) - E'(k)]\right]. \tag{22}$$

For $\nu\geqq 4/\pi$ there are no functions in Σ which can be obtained from $f_\nu(z)$ by translation along the trajectories of $Q_\nu(w)dw^2$.

The upper half w-plane is carried by the mapping

$$\omega = \int w^{-1}(w^2 - \nu^2)^{1/2}dw$$

for a suitable choice of determination onto the domain bounded by the following half-infinite segments: $\mathcal{I}\omega=0$, $\mathcal{R}\omega\geqq\pi\nu/2$; $\mathcal{I}\omega<0$, $\mathcal{R}\omega=\pi\nu/2$; $\mathcal{I}\omega<0$, $\mathcal{R}\omega=-\pi\nu/2$; $\mathcal{I}\omega=0$, $\mathcal{R}\omega\leqq-\pi\nu/2$. The explicit form of the mapping is

$$\omega = (w^2 - \nu^2)^{1/2} - \frac{1}{2}i\nu\log\frac{(w^2 - \nu^2)^{1/2} + i\nu}{(w^2 - \nu^2)^{1/2} - i\nu}$$

where $(w^2-\nu^2)^{1/2}$ is to have its positive determination for w large and positive and the logarithm is to have its positive determination at the same points.

We take

$$\zeta = z + z^{-1}. \tag{23}$$

Then provided $\pi\nu/2\leqq 2$, setting

$$\omega = \zeta$$

we obtain an induced mapping from E^* into the w-plane which is just the desired mapping f_ν. It has expansion about the point at infinity

$$f_\nu(z) = z + \frac{1}{2}(2 - \nu^2)/z + \cdots.$$

The expansion of the mapping ω about the origin is given by

$$\omega = i\nu \log w + i\nu - \frac{1}{2} i\nu \log 4\nu^2 + \frac{1}{2}\pi\nu + \text{positive powers of } w.$$

Thus the circle $|z| < 1$ is mapped conformally on the interior of the complement of $f_\nu(E^*)$ so that $z=0$ goes into $w=0$ by the mapping $w=g_\nu(z)$ induced by setting

$$i\nu \log z + \frac{1}{2}\pi\nu = \omega.$$

Consequently the inner conform radius with respect to the origin of the above domain is $2\nu e^{-1}$.

A translation along the trajectories of $Q_\nu(w)dw^2$ in the w-plane corresponds to a horizontal translation in the ω-plane. This is consistent with the preceding construction provided the amount s of translation does not exceed the length in the Q-metric of each of the slits (present if $\pi\nu/2 < 2$) on the boundary of $f_\nu(E^*)$ on the real axis. This requires $-(2-\pi\nu/2) \leqq s \leqq 2-\pi\nu/2$. Then we obtain the new function $F_\nu(z, s)$ by setting

$$\omega = \zeta + s$$

so that the expansion of $F_\nu(z, s)$ about the point at infinity begins as in (20).

When $\pi\nu/2 > 2$ we use instead of (23) the mapping

$$\zeta = \int z^{-2}[(z^2 - \lambda^2)(z^2 - \lambda^{-2})]^{1/2} dz \tag{24}$$

of $E^* \cap \{\mathfrak{I}z > 0\}$ where the root and the constant of integration are chosen so that ζ is positive for z large and positive and the mapping is symmetric in the respective imaginary axes. Making the substitution $Z = (z+z^{-1})/2$ we obtain

$$\zeta = \frac{2}{k}\int \left(\frac{1 - k^2Z^2}{1 - Z^2}\right)^{1/2} dZ$$

where

$$k = \frac{2\lambda}{1 + \lambda^2}$$

and the mapping satisfies the same conditions as before. Then provided

$$\frac{1}{2}\pi\nu = \frac{2}{k}E(k)$$

setting

$$\omega = \zeta$$

the induced mapping from E^* into the w-plane will provide just the desired mapping f_ν. Its expansion about the point at infinity is

$$z + \left(2k^{-2} - 1 - \frac{1}{2}\nu^2\right)z^{-1} + \textit{higher powers of } z^{-1}.$$

In this case the circle $|z| < 1$ is mapped conformally on the interior of the complement of $f_\nu(E^*)$ so that $z=0$ goes into $w=0$ by the mapping $w=g_\nu(z)$ induced by setting

$$i\nu \log z + \frac{1}{2}\pi\nu - i\frac{2}{k}[K'(k) - E'(k)] = \omega.$$

Thus the inner conform radius with respect to the origin of the above domain is

$$2\nu e^{-1} \exp\left[-2k^{-1}\nu^{-1}[K'(k) - E'(k)]\right].$$

That for $\nu \geqq 4/\pi$ no functions in Σ can be obtained from $f_\nu(z)$ by translation along the trajectories of $Q_\nu(w)dw^2$ is readily seen by examining these trajectories in the neighborhood of $w=\nu$ (or $-\nu$).

THEOREM 2. *Let $f \in \Sigma$ map E^*: $|z| > 1$ onto a domain whose complement contains a domain with inner conform radius with respect to the origin at least r $(0 < r < 1)$ and have expansion about the point at infinity*

$$z + a_0 + a_1 z^{-1} + \textit{higher powers of } z^{-1}.$$

Then the region of possible values of a_1 is given by

$$|a_1| \leqq \mathrm{P}_r$$

where

$$\mathrm{P}_r = 1 - \frac{1}{8}e^2 r$$

for $r \leqq 8/\pi e$ and

$$\mathrm{P}_r = 2k^{-2} - 1 - \frac{1}{2}\nu^2$$

with

$$(25) \qquad \nu = \frac{4}{\pi k} E(k), \qquad r = 2\nu e^{-1}[-2k^{-1}\nu^{-1}[K'(k) - E'(k)]]$$

for $r > 8/\pi e$. *For* $r < 8/\pi e$ *the value* $P_r e^{2i\theta}$, θ *real*, $0 \leqq \theta < \pi$, *is attained only for the functions* $e^{i\theta}F_\nu(e^{-i\theta}z, s)$ *where* $\nu = er/2, -(2-\pi\nu/2) \leqq s \leqq 2 - \pi\nu/2$. *For* $r \geqq 8/\pi e$ *the value* $P_r e^{2i\theta}$, θ *real*, $0 \leqq \theta < \pi$, *is attained only for the function* $e^{i\theta}f_\nu(e^{-i\theta}z)$ *where* ν *is determined by* (25).

Let E denote $|z| < 1$ and let $\Phi_\nu(w)$, $\Psi_\nu(w)$ be the respective inverses of f_ν, g_ν defined on $f_\nu(E^*)$, $g_\nu(E)$. Corresponding to a given function $f(z)$ there exists $g(z)$ mapping E conformally into the complement of $f(E^*)$ with $g(0) = 0$, $g'(0) = r$. For given real θ, $0 \leqq \theta < \pi$, we apply Theorem 1 with $\mathfrak{R}$ the w-sphere, the quadratic differential

$$Q(w)dw^2 = e^{-2i\theta}(w^2 - \nu^2 e^{2i\theta})dw^2/w^2,$$

the admissible family of domains $e^{i\theta}f_\nu(E^*)$, $e^{i\theta}g_\nu(E)$ and the admissible family of functions $f(e^{i\theta}\Phi_\nu(e^{-i\theta}w))$, $g(e^{i\theta}\Psi_\nu(e^{-i\theta}w))$.

The quadratic differential has a double pole P_1 at the origin and a pole P_2 of order four at the point at infinity. We have $m_2 = 4$, $k_2 = 0$. The corresponding coefficients are

$$\alpha^{(1)} = -\nu^2, \qquad a^{(1)} = g_\nu'(0)/g'(0),$$

$$\alpha^{(2)} = e^{-2i\theta}, \qquad \beta_1^{(2)} = 0, \qquad a_0^{(2)} = a_0, \qquad a_1^{(2)} = a_1 - e^{2i\theta}P_r.$$

Inequality (6) then gives

$$\mathfrak{R}\{-\nu^2 \log (g_\nu'(0)/g'(0)) + e^{-2i\theta}(a_1 - e^{2i\theta}P_r)\} \leqq 0$$

or

$$\mathfrak{R}\{e^{-2i\theta}a_1\} \leqq P_r, \qquad 0 \leqq \theta < \pi,$$

that is

$$|a_1| \leqq P_r.$$

That the value $P_r e^{2i\theta}$ can be attained only for the functions indicated follows from the fact that they are the only functions obtained from $e^{i\theta}f_\nu(e^{-i\theta}z)$ by translation along the trajectories of $Q(w)dw^2$. That the circular disc is the exact region of possible values of a_1 follows by a continuity argument closely related to Grötzsch's argument [2, p. 94].

It is worth while pointing out what advantage has been gained in using the present Theorem 1 to solve this problem rather than the earlier form of the General Coefficient Theorem. To treat the problem by the latter method it would have been necessary to find the best possible bound for $|a_1|$ for each possible value of a_0 and then solve explicitly the calculus problem so obtained. There are numerous other interesting applications of this extended form of the General Coefficient Theorem some of which may be treated in later publications.

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